Anti-pluricanonical systems on Fano varieties.

Preliminaries

Defn: Let X be a variety. A **b-$\mathbb{R}$-Cartier b-divisor over X** is the choice of $Y \to X$ projective birational with Y normal variety, and M an $\mathbb{R}$-Cartier divisor on Y, up to the equivalence

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ M \subset Y \arrow[dr] & & Y' \supset M' \arrow[dl] \\ & X & \end{tikzcd} \quad ; \qquad p^* M = q^* M'$$

Defn: Let (X, B) be a pair and $X \to Z$ a contraction. We say (X,B) is **log Fano over Z** if it's lc and $-(K_X+B)$ is ample over Z, and we say it's **weak log Fano over Z** if $-(K_X+B)$ is big and nef over Z.

We say X is of **Fano type over Z** if (X,B) is klt log Fano $/Z$ for some choice of B. This is equivalent to the existence of Γ, a big $/Z$ $\mathbb{Q}$-boundary so that (X, Γ) is klt and

$$K_X + \Gamma \sim_{\mathbb{Q}} 0 / Z.$$

Lemma 2.11: Let (X,B) be an lc pair and $f: X \to Z$ be a contraction, with Z a smooth curve. Assume X is of Fano type $U \subseteq Z$ open. Also assume B is a $\mathbb{Q}$-boundary, $K_X + B \sim_{\mathbb{Q}} 0/Z$, and the generic point of each non-klt center of (X,B) is mapped into U. Then X is of Fano type.

Proof:

- Take Γ a $\mathbb{Q}$-boundary such that Γ is big$/U$, (X,Γ) is klt over U, and $K_X + \Gamma \sim_{\mathbb{Q}} 0/U$.
- After some perturbation, we can assume that $K_X + \Gamma \sim_{\mathbb{Q}} D/Z$, with D vertical$/Z$, $D \leq 0$ and the support of D is mapped into $Z \setminus U$.
- The generic point of non-klt center of (X,B) is mapped into U + (X,Γ) is klt $/U$

$$\Downarrow$$

$(X, (1-t)B + t(\Gamma - D))$ is klt for $0 < t \ll 1$.

- $K_X + (1-t)B + t(\Gamma - D) = (1-t)\underbrace{(K_X + B)}_{\sim_{\mathbb{Q}} 0/Z} + t\underbrace{(K_X + \Gamma - D)}_{\sim_{\mathbb{Q}} 0/Z} \sim_{\mathbb{Q}} 0$

+

$(1-t)B + t(\Gamma - D)$ is big$/Z$

$\Rightarrow$ X is Fano type $/Z$

Lemma 2.12: Let X be a projective variety of Fano type, and let $f: X \to Z$ be a contraction with $\dim Z > 0$. Then Z is Fano type.

Proof:

- X normal $\Rightarrow$ Z normal
- We have Γ big $\mathbb{Q}$-boundary with $K_X+\Gamma \sim_{\mathbb{Q}} 0$ klt. We can assume $\Gamma \geq H \geq f^*A$, with H ample, and A ample on Z, and set $\Delta := \Gamma - f^*A$.
- $K_X + \Delta \sim_{\mathbb{Q}} 0/Z \Rightarrow \exists\, \Delta_Z$ such that $K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Z + \Delta_Z)$ and (Z, Δ_Z) is klt. [Ambro '05]
- After perturbing Δ_Z, we obtain Γ_Z big, with (Z, Γ_Z) klt and $K_Z + \Gamma_Z \sim_{\mathbb{Q}} 0$.

Defn: A generalized pair consists of

- A normal variety X' with a projective morphism $X' \longrightarrow Z$,
- An $\mathbb{R}$-divisor $B' \geq 0$ on X,
- A b-$\mathbb{R}$-Cartier divisor over X' represented by some projective birational morphism $X \xrightarrow{\phi} X'$ and $\mathbb{R}$-Cartier divisor M on X, such that M is nef$/Z$ and $K_{X'} + B' + M'$ is $\mathbb{R}$-Cartier, with $M' := \phi_* M$.

- We can assume $X \xrightarrow{\phi} X'$ is a log resolution of (X', B) and write

$$K_X + B + M = \phi^*(K_{X'} + B' + M')$$

For a prime divisor D on X define the generalized log discrepancy $a(D, X', B'+M')$ as $1 - \mu_D B$.
We say $(X', B'+M')$ is generalized lc (klt) (ε-lc) if $a(D, X', B'+M') \geq 0$ (>0) $(\geq \varepsilon)$ for each D.
A generalized non-klt center of $(X', B'+M')$ is the image of a prime divisor D on a birational model of X' with $a(D, X', B'+M') \leq 0$.

We say $(X', B'+M')$ is *generalized dlt* if it is gen. lc and if η is the generic point of any gen non-klt center of $(X', B'+M')$, then (X', B') is log smooth near η and $M = \phi^* M'$ holds near η. The gen. dlt property is preserved under the MMP.

Let $\psi: X'' \to X'$ be a projective birational morphism. We can assume

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X'' \arrow[rr, "\psi"] & & X' \end{tikzcd}$$

If $B'' \geq 0$, then $(X'', B''+M'')$ is also a gen. pair.
If $(X'', B''+M'')$ is $\mathbb{Q}$-factorial and every exc. prime divisor of ψ appears in B'' with coefficient 1, then $(X'', B''+M'')$ is a *$\mathbb{Q}$-factorial generalized dlt model* of $(X', B'+M')$.

Lemma 2.14: Let $(X', B'+M')$ be a generalized pair, with $X \xrightarrow{\phi} X' \to Z$ where $X' \to Z$ is a contraction. Assume $-(K_{X'}+B'+M')$ is nef and big $/Z$. Then the gen. non-klt locus of $(X', B'+M')$ is connected near each fiber of $X' \to Z$.

Proof:

- Assume ϕ is a log resolution, and write
$$K_X + B + M = \phi^*(K_{X'} + B' + M').$$
Write $-(K_X + B + M) \sim_{\mathbb{R}} A + C / Z$, with A ample and $C \geq 0$.
- Assume now that ϕ is a log resolution of $(X', B'+C')$.
- Pick $0 < \varepsilon \ll 1$, and let $G \sim_{\mathbb{R}} M + \varepsilon A / Z$ be general w/ coefficients less than 1, and let $\Delta = B + \varepsilon C + G$.
- $K_X + \Delta \sim_{\mathbb{R}} 0 / X'$, so $K_X + \Delta = \phi^*(K_{X'} + \Delta')$ and also $\operatorname{Supp} B^{\geq 1} = \operatorname{Supp} \Delta^{\geq 1}$.
- Finally $-(K_{X'} + \Delta') \sim_{\mathbb{R}} -(1-\varepsilon)(K_{X'} + B' + M')/Z$ is big + nef.

Same trick proves that if $-(K_{X'}+B'+M')$ is nef and big, then X' is of Fano type.

Defn: Let $(X', B'+M')$ be a proj. gen. pair with $X \xrightarrow{\phi} X'$ and M. Assume $K_{X'}+B'+M'+P' \sim_{\mathbb{R}} 0$ for some $\mathbb{R}$-divisor $P' \geq 0$. We say the pair is *non-exceptional* if we can choose P' so that $(X', B'+M'+P')$ is not gen. klt, and *exceptional* if not.

Lemma 2.17: Let $(X', B'+M')$ and $(X'', B''+M'')$ be proj. gen. pairs with data

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ X' & & X'' \end{tikzcd}$$

and M.

Assume $\psi^*(K_{X''}+B''+M'') \geq \phi^*(K_{X'}+B'+M')$. If $(X', B'+M')$ is exceptional, then $(X'', B''+M'')$ is also exceptional.

Defn: Let $(X', B'+M')$ be a proj. gen. pair, where $B' \in [0,1]$. Let $T' = \lfloor B' \rfloor$ and $\Delta' = B'-T'$. An *n-complement* of $K_{X'}+B'+M'$ is of the form $K_{X'}+B'^{+}+M'$, where

- $(X', B'^{+}+M')$ is gen. lc;
- $n(K_{X'}+B'^{+}+M') \sim 0$ and nM is b-Cartier;
- $nB'^{+} \geq nT' + \lfloor (n+1)\Delta' \rfloor$.

The definition implies that $|\underbrace{-nK_{X'}-nM'}_{\sim nB'^{+}} - nT' - \lfloor (n+1)\Delta' \rfloor|$ is not empty.

Same for $|-n(K_{X'}+B'+M')|$ if $B'^{+} \geq B'$.

Defn: A *couple* (X, D) consists of a normal proj. variety X and a reduced divisor D. We say (X, D) and (X', D') are isomorphic if there is an iso $X \xrightarrow{\sim} X'$ that sends D onto D'.

We say that a set $\mathfrak{P}$ of couples is **birationally bounded** if there exist finitely many projective morphisms $V^i \to T^i$ of varieties and reduced divisors C^i on V^i s.t. for each $(X, D) \in \mathfrak{P}$ there exist an i, a closed point $t \in T^i$, and a birational isomorphism $\phi: V^i_t \dashrightarrow X$ s.t. (V^i_t, C^i_t) is a couple and $E \leq C^i_t$, with E the sum of the bir. transform of D and the reduced exceptional divisor of ϕ.
If ϕ is an isomorphism we say $\mathfrak{P}$ is **bounded**.

Lemma 2.24: Let $\mathfrak{P}$ be a bounded set of couples. Then there is a natural number I depending only on $\mathfrak{P}$ satisfying the following. Assume X is projective w/ klt sings. and $M \geq 0$ an integral $\mathbb{Q}$-Cartier divisor on X so that $(X, \operatorname{Supp} M) \in \mathfrak{P}$. Then IK_X and IM are Cartier.

Defn: Let X be a proj variety. A bounded family $\mathcal{G}$ of subvarieties of X is a family of subvarieties such that there are finitely many morphisms $V^i \to T^i$ of proj varieties together with morphisms $V^i \to X$ such that it embeds in X the fibers of $V^i \to T^i$ over closed points, and each member of $\mathcal{G}$ is isomorphic to a fiber of some $V^i \to T^i$ for some closed points.

We say that $\mathcal{G}$ is a covering family of subvarieties of X if the union of its members contains some non-empty open subset of X. We say $G \in \mathcal{G}$ is a general member of $\mathcal{G}$ if some $V^i \to X$ is surjective, the set of points of T^i corresponding to $\mathcal{G}$ are dense in T^i, and G is a fiber over a general point.

Lemma 2.28: Let $f: V \to T$ be a contraction between smooth proj varieties and $g: V \to X$ a surjective morphism to a normal proj. variety. Let t be a closed point of T and F the fiber of f over t. Assume:

(1) the induced map $F \to X$ is birational onto its image,

(2) f is smooth over t,

(3) g is smooth over $g(\eta_F)$, and $g(\eta_F)$ is a smooth point of X.

Let S be a general hypersfc section of T of large degree passing through t, let $U = f^*S$, and assume $U \to X$ is surjective. Then U and S are smooth, $U \to S$ is smooth over t, and $U \to X$ is smooth over $g(\eta_F)$.

Lemma 2.32: Let (X,B) be a proj. pair where B is a $\mathbb{Q}$-boundary, and let $D\geq 0$ be an ample $\mathbb{Q}$-divisor. Let $x,y\in X$ be closed points, and assume (X,B) is klt near x, $(X,B+D)$ is lc near x with a unique non-klt center G containing x, and $(X,B+D)$ is not klt near y. Then there exist rational numbers $0\leq t \ll s \leq 1$ and a $\mathbb{Q}$-divisor $0\leq E\sim_{\mathbb{Q}} tD$ s.t the $(X,B+sD+E)$ satisfies the above condition, and there is a unique non-klt place over G.

Application: Let X be normal proj variety of dim d and D ample $\mathbb{Q}$-divisor. Assume $\operatorname{vol}(D)>(2d)^d$. Then there is a bounded family of subvarieties of X s.t. for each pair $x,y\in X$ of general closed points, there is a member G of the family and $0\leq \Delta\sim_{\mathbb{Q}} D$ such that (X,Δ) is lc near x w/ unique non-klt place whose center is $G\ni x$, and (X,Δ) is not klt at y.

- Assume that A is ample effective. Pick $x,y\in X$ general and Δ and G as above. If $\dim G=0$ or if $\dim G>0$ and $\operatorname{vol}(A|_G)\leq d^d$, then let $G':=G$ and let $\Delta':=\Delta+A$. If not, there is $0\leq\Delta'\sim_{\mathbb{Q}}\Delta+A$ and a proper subvar $G'\subset G$ s.t. (X,Δ') has the same properties as above, with non-klt center G'.

We repeat this procedure $d-1$ times to obtain $0\leq \Delta^{(d-1)}\sim_{\mathbb{Q}} D+(d-1)A$ and $G^{(d-1)}\subsetneq G$ s.t. the above condition holds, and either $\dim G^{(d-1)}=0$ or $\operatorname{vol}(A|_{G^{(d-1)}})\leq d^d$. In particular, the centers $G^{(d-1)}$ form a bounded family of subvars.